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On the Proof of Universality for Orthogonal and Symplectic Ensembles in Random Matrix Theory

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We give a streamlined proof of a quantitative version of a result from P. Deift and D. Gioev, *Universality in Random Matrix Theory for Orthogonal and Symplectic Ensembles*. *IMRP Int. Math. Res. Pap.* (in press) which is crucial for the proof of universality in the bulk P. Deift and D. Gioev, *Universality in Random Matrix Theory for Orthogonal and Symplectic Ensembles*. *IMRP Int. Math. Res. Pap.* (in press) and also at the edge P. Deift and D. Gioev, Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices. *Comm. Pure Appl. Math.* (in press) for orthogonal and symplectic ensembles of random matrices. As a byproduct, this result gives asymptotic information on a certain ratio of the $\beta = 1, 2, 4$ partition functions for log gases.

KEY WORDS: random matrix theory, universality, orthogonal and symplectic ensembles, partition function, log gases

For $m \geq 2$, let

$$h(x) = \sum_{k=0}^{m-1} \beta_k x^{2k}$$

$$\beta_k = 2\frac{(2m)(2m-2)\cdots(2m-2k)}{(2m-1)(2m-3)\cdots(2m-2k-1)}, \qquad 0 \leq k \leq m-1. \quad (1)$$

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For odd q set

$$I(q) \equiv \frac{2}{\pi} \sin \frac{q\pi}{2} \int_{-1}^{1} \frac{\cos(q \arcsin x)}{h(x)(1-x^2)}\, dx = \frac{4}{\pi} \int_{0}^{\pi/2} \frac{\sin qs}{\sin s \; h(\cos s)} ds \tag{2}$$

and

$$Q(q) \equiv I(q) + \frac{1}{2m}. \tag{3}$$

For $n \equiv 2m-1$, define the $(m-1)\times(m-1)$ matrix

$$T^{[m-1]} = I - \frac{(m!)^2}{m(2m)!} Q^{[m-1]} B^{[m-1]} \equiv I - K^{[m-1]} \tag{4}$$

where

$$Q_{ij}^{[m-1]} = Q(n-2i+2j), \qquad B_{ij}^{[m-1]} = 2m \binom{n}{j-i}, \qquad 1 \le i, j \le m-1.$$

Here $\binom{n}{k} \equiv 0$ for $k < 0$.

In (Ref. 2, Theorem 2.6), the authors prove the following result: for $m \ge 2$,

$$\det T^{[m-1]} \ne 0 \tag{5}$$

(see Remark 2 after the proof of Theorem 1 below). Note that in the notation of Ref. 2, $T^{[m-1]} = T_{m-1}$.

In this paper we will give a streamlined proof of the following quantitative version of (5).

Theorem 1. *For $m \ge 2$,*

$$\det T^{[m-1]} \ge 0.0865. \tag{6}$$

Equation (5) plays a crucial role in proving universality in the bulk,[2] and also at the edge,[3] for orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) random matrix ensembles for a class of weights $w(x) = e^{-V(x)}$ where $V(x)$ is a polynomial $V(x) = \kappa_{2m} x^{2m} + \cdots, \kappa_{2m} > 0$. (Here m is the same integer as in (5) and (6).) The situation is as follows. In Refs. 2, 3, and also in Ref. 4, the authors use the method of Widom,[11] which is based in turn on Ref. 10, together with the asymptotic analysis for orthogonal polynomials in Ref. 5. A new and challenging feature of the method in Ref. 11, which does not arise in the proof of universality in the case $\beta = 2$, is the appearance of the *inverse* of a certain matrix C_{11} of fixed size $n = 2m-1$ (see Ref. 2, (1.37) and Theorem 2.3 et seq.). In the scaling limit as $N \to \infty$, the matrix C_{11} converges to a matrix C_{11}^{∞} and

$$\det C_{11}^{\infty} = \left(\det T^{[m-1]}\right)^2 \tag{7}$$

(see discussion from (2.13) up to Theorem 2.4 in Ref. 2). Thus in order to control the scaling limit for $\beta = 1$ and 4, we need to show that $\det T^{[m-1]} \neq 0$.

It turns out that $\det T^{[m-1]}$ is related to partition functions for finite log gases in an external field V at inverse temperatures $\beta = 1, 2, 4$

$$Z_{V,\beta,k} \equiv \frac{1}{k!} \int \cdots \int \prod_{1 \le i < j \le k} |x_i - x_j|^\beta e^{-\sum_{i=1}^k V(x_i)}\, dx_1 \ldots dx_k$$
$$= \frac{1}{k!} \int \cdots \int e^{-\beta \sum_{1 \le i<j\le k} \log|x_i - x_j| - \sum_{i=1}^k V(x_i)}\, dx_1 \ldots dx_k. \tag{8}$$

Using standard formulae for such partition functions (see e.g. Ref. 1, (4.4), (4.17), (4.20)), together with (Ref. 2, (2.18)), one finds (see Ref. 9, Remark 2.4, Ref. 2, Remark 1.5) that for ensembles of (even) size N

$$\det C_{11} = \left(\frac{1}{2^N (N/2)!} \frac{Z_{2V,4,N/2}\, Z_{V,1,N}}{Z_{2V,2,N}} \right)^2. \tag{9}$$

Thus

$$\lim_{N\to\infty} \frac{Z_{2V,4,N/2}\, Z_{V,1,N}}{2^N (N/2)!\, Z_{2V,2,N}} = \det T^{[m-1]} \neq 0. \tag{10}$$

Formula (9), together with (7), raises the possibility of using the methods of statistical mechanics to prove (5), (6). The estimates in Ref. 7 show that the partition functions $Z_{V,\beta,k}$ have, for certain constants $\alpha_{V,\beta}$, leading order asymptotics of the form $e^{\alpha_{V,\beta} k^2 (1+o(1))}$ as $k \to \infty$, and moreover, their combined contributions to $\det C_{11}$ cancel to this order. In order to achieve cancellation at subsequent orders, and so prove (5), (6), one needs higher order asymptotics for the $Z_{V,\beta,k}$'s, but, unfortunately such asymptotics are known only for $\beta = 2$ (see Ref. 6). Regarding (9), we take the contrary point of view, i.e., (10) and (6) provide new information on the asymptotics of partition functions for log gases at inverse temperatures $\beta = 1$ and 4.

Much of the analysis in Ref. 2 involves estimating $Q(q)$ in two regions: $3 \le q \lesssim \sqrt{m}$ and $\sqrt{m} \lesssim q \le 4m - 5$. In this note, using bounds on

$$W(x) \equiv \frac{2}{\pi} \int_0^x \frac{\sin qs}{\sin s}\, ds \tag{11}$$

which are uniform in $q = 3, 5, \ldots$ and in $0 \le x \le \pi/2$ (see Lemma 4 below), we are able to estimate $Q(q)$ uniformly for $q = 3, 5, \ldots, 4m - 5$ and so avoid many of the technicalities in the proof in Ref. 2 of (5). Of course the function $W(x)$ is familiar from the analysis of the Gibbs phenomenon in Fourier analysis.

Remark 1. For $m = 1$, corresponding to the Gaussian orthogonal and symplectic ensembles, $T^{[m-1]}$ is not defined and no analog of (5), (6) is needed (see Ref. 2).

We use the following result. For a matrix X let $r(X) = \sup\{|\lambda| : \lambda \in \text{spec}X\}$ denote the spectral radius of X. As is well known, for any operator norm $\|\cdot\|$ on $\{X\}$,

$$r(X) = \lim_{j\to\infty} \|X^j\|^{1/j} = \inf_{j\geq 1} \|X^j\|^{1/j}. \tag{12}$$

Lemma 2. *Assume K and K' are J-dimensional matrices with real entries such that $|K_{ij}| \leq K'_{ij}$, $1 \leq i, j \leq J$, and $r(K') < 1$. Then $r(K) < 1$ and*

$$\det(I - K) \geq \det(I - K') > 0. \tag{13}$$

Proof: The following is true: if $r(X) < 1$, then

$$\det(I - X) = e^{-\sum_{l=1}^{\infty} \frac{1}{l}\ \text{tr}(X^l)}. \tag{14}$$

This result is usually stated in the form that (14) holds if $\|X\| < 1$ (see e.g. Ref. 8). To obtain (14) for $r(X) < 1$ from the case $\|X\| < 1$ simply apply (14) to μX for μ small and observe that for any fixed ρ satisfying $r(X) < \rho < 1$, $\|X^l\| \leq \rho^l$ for l sufficiently large: then (14) follows for $r(X) < 1$ by analytic continuation $\mu \to 1$.

Equip $\mathbb{R}^J$ with the l_∞ -norm $\|\cdot\|_\infty$ (any l_p-norm, $1 \leq p \leq \infty$ would do) and for a matrix X mapping $\mathbb{R}^J \to \mathbb{R}^J$ denote the associated operator norm by $\|X\|$. For $\phi = \{\phi_j\} \in \mathbb{R}^J$ we denote the vector with coordinates $\{|\phi_j|\}$ by $|\phi|$. We claim that $r(K) \leq r(K')$. Indeed, for $\phi \in \mathbb{R}^J$, $|(K^l\phi)_j| \leq ((K')^l|\phi|)_j$ and so

$$\|K^l\phi\|_\infty \leq \|\,(K')^l|\phi|\,\|_\infty \leq \|(K')^l\|\ \|\,|\phi|\,\|_\infty = \|(K')^l\|\ \|\phi\|_\infty.$$

Thus $\|K^l\| \leq \|(K')^l\|$ and so $r(K) \leq r(K') < 1$ by (12). It follows that (14) is valid for K and K'. But clearly $|\text{tr}(K^l)| \leq \text{tr}((K')^l)$ and (13) is now immediate. □

The function $h(x)$ in (1) has the following properties (see Ref. 2, Proposition 6.2): for $0 < x < 1$

$$\begin{aligned}
&\text{(i)} \quad h \text{ solves the differential equation} \\
&\qquad x(x^2 - 1)h' + (2m - 1 - 2(m-1)x^2)h = 4m \\
&\text{(ii)} \quad \frac{4m}{2m-1} = h(0) \leq h(x) \leq h(1) = 4m \\
&\text{(iii)} \quad h(x) = \frac{4mx^{2m-1}}{\sqrt{1-x^2}} \int_x^1 \frac{t^{-2m}}{\sqrt{1-t^2}}\, dt.
\end{aligned} \tag{15}$$

Property (i) reflects the fact that h is a hypergeometric function,

$$h(x) = \frac{4m}{2m-1}{}_2F_1(1, -m+1, -m+3/2; x^2)$$

(see Ref. 2, (6.11)) and (iii) follows by integrating (i). Property (ii) follows from (i) and (1).

Set

$$u(x) \equiv u(x; m) = \frac{1}{h(x)} - \frac{1-x^2}{2} + \frac{1}{4m}. \tag{16}$$

Note that the function $u(x)$ is closely related to the function y_m which plays a prominent role in Ref. 2: we have

$$u(x) = \frac{\sqrt{1-x^2}}{m} y_m(\arcsin x) + \frac{1}{2m}, \qquad 0 \le x \le 1.$$

Also note that using the elementary identities for $q = 3, 5, \ldots$,

$$\frac{2}{\pi}\int_0^{\pi/2} \sin qs\ \sin s\, ds = 0, \qquad W\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sin qs}{\sin s}\, ds = 1$$

we have from (2), (16)

$$I(q) = \frac{4}{\pi}\int_0^{\pi/2} \frac{\sin qs}{\sin s} u(\cos s)\, ds - \frac{1}{2m}. \tag{17}$$

The main technical result in our proof of Theorem 1 is the following.

Lemma 3. *The function $u(x) = u(x; m)$, $m \ge 2$, has the following properties.*

(i) *$u(x)$ is unimodal for $x \in [0, 1]$. More precisely, there exists $x_0 \in (0, 1)$ such that $u'(x) < 0$ for $0 < x < x_0$ and $u'(x) > 0$ for $x_0 < x < 1$.*

(ii) *$u(0) = 0$, $u(1) = \frac{1}{2m}$.*

(iii) *For $0 \le x \le 1$,*

$$-\frac{1}{4m} < u(x) \le \frac{1}{2m}.$$

The proof of Lemma 3 is given after the proof of Theorem 1 below. We also need the following elementary result from Fourier analysis.

Lemma 4. *For $q \ge 3$, $0 \le x \le \pi/2$,*

$$0 \le W(x) \le \frac{\sqrt{3}}{\pi} + \frac{2}{3} < 1.218.$$

Proof: As the factor $\sin s$ in $W(x) = \frac{2}{\pi}\int_0^x \frac{\sin qs}{\sin s}\, ds$ is increasing, a standard argument in the analysis of the Gibbs phenomenon shows that for $0 \le x \le \pi/2$, $0 \le W(x) \le \frac{2}{\pi}\int_0^{\pi/q} \frac{\sin qs}{\sin s}\, ds = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{q \sin(t/q)}\, dt$. But for $0 \le t \le \pi/2$, $q \mapsto q \sin(t/q)$ is increasing, and so for $q \ge 3$ and $0 \le x \le \pi/2$, $0 \le W(x) \le \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{3\sin(t/3)}\, dt = \frac{\sqrt{3}}{\pi} + \frac{2}{3}$. □

Assuming Lemma 3, we now prove Theorem 1. By (3), (17), integrating by parts and using Lemma 3(ii),

$$\begin{aligned}
Q(q) &= \frac{4}{\pi}\int_0^{\pi/2} \frac{\sin qs}{\sin s}\, u(\cos s)\, ds \\
&= 2\big(W(s)\, u(\cos s)\big)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} W(s)\, u'(\cos s)\ \sin s\, ds \\
&= 2\bigg(\int_0^{\arccos x_0} + \int_{\arccos x_0}^{\pi/2}\bigg) W(s)\, u'(\cos s)\ \sin s\, ds.
\end{aligned}$$

Thus, by Lemma 3 and Lemma 4,

$$\begin{aligned}
Q(q) &\le 2\int_0^{\arccos x_0} W(s)\, u'(\cos s) \sin s\, ds \\
&\le 2 \times 1.218\big(u(1) - u(x_0)\big) \\
&\le 2 \times 1.218\Big(\frac{1}{2m} + \frac{1}{4m}\Big) = \frac{3}{2}\frac{1.218}{m}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
Q(q) &\ge 2\int_{\arccos x_0}^{\pi/2} W(s)\, u'(\cos s) \sin s\, ds \\
&\ge 2 \times 1.218\big(u(x_0) - u(0)\big) \ge -\frac{1}{2}\frac{1.218}{m}
\end{aligned}$$

and thus

$$|Q(q)| \le \frac{3}{2}\frac{1.218}{m} = \frac{1.827}{m}.$$

Recalling the definitions of $Q^{[m-1]}$ and $B^{[m-1]}$, we have for $1 \le i, j \le m-1$

$$\big|\big(Q^{[m-1]}B^{[m-1]}\big)_{ij}\big| \ \le \Big|\sum_{l=1}^{j} Q(n-2i+2l)\, 2m \binom{n}{j-l}\Big| \le 2 \times 1.827 \sum_{l=0}^{j-1}\binom{n}{l}$$

and hence

$$\big|\big(Q^{[m-1]}B^{[m-1]}\big)_{ij}\big| \le 2 \times 1.827 L_{ij}, \qquad 1 \le i, j \le m-1,$$

where L is the rank 1 matrix with entries $L_{ij} = \sum_{l=0}^{j-1}\binom{n}{l}$, independent of i.

Hence L has only 1 non-zero eigenvalue $\lambda_1(L)$ and we find

$$r(L) = \lambda_1(L) = \sum_{k=1}^{m-1} L_{1k} = \sum_{k=1}^{m-1}\sum_{l=1}^{k} \binom{n}{l-1}$$

$$= \sum_{l=0}^{m-1}(m-l-1)\binom{2m-1}{l} = \frac{m}{2}\binom{2m-1}{m-1} - 2^{2m-3} \leq \frac{m}{2}\binom{2m-1}{m-1}. \tag{18}$$

In the second last step, we have used the elementary formula preceding (6.7) in Ref. 2.

Assembling the above results and recalling the definition of $K^{[m-1]}$, we obtain for $1 \leq i, j \leq m-1$,

$$\left|K_{ij}^{[m-1]}\right| = \frac{(m!)^2}{m(2m)!}\left|\left(Q^{[m-1]}B^{[m-1]}\right)_{ij}\right| \leq K'_{ij}$$

where

$$K'_{ij} \equiv 2 \times 1.827 \frac{(m!)^2}{m(2m)!} L_{ij}, \qquad 1 \leq i, j \leq m-1,$$

and by (18), the only non-zero eigenvalue of K' satisfies

$$\lambda_1(K') = r(K') = 2 \times 1.827 \frac{(m!)^2}{m(2m)!} r(L)$$

$$\leq 2 \times 1.827 \frac{(m!)^2}{m(2m)!} \frac{m}{2}\binom{2m-1}{m-1} = \frac{1.827}{2} = 0.9135 < 1. \tag{19}$$

Thus by Lemma 2,

$$\det\left(1 - K^{[m-1]}\right) \geq \det(1 - K') = 1 - \lambda_1(K') \geq 0.0865.$$

This completes the proof of Theorem 1.

Remark 2. Using Lemma 2, the calculations in Ref. 2 also yield a quantitative version of (5) but with a weaker bound. As above, we estimate $T^{[m-1]}$ elementwise with a rank one matrix so that we can estimate the determinant by estimating the only nonzero eigenvalue. We note that we cannot use (Ref. 2, (6.22)) (the matrix in (6.22) is not rank one). For "small" m we use (Ref. 2, (6.15), (6.16)), and for "large" m we use (Ref. 2, (6.55), (6.21)). We claim that

$$\det T^{[m-1]} \geq 0.02, \qquad m \geq 2. \tag{20}$$

This estimate is not optimal, but we could not strengthen it compared to (6) by the methods in Ref. 2. To prove (20) for $2 \leq m \leq 46$, we note that the RHS in (Ref. 2, (6.16)) is < 0.98 for m in this range. (Note that our $Q(q)$ and $\tilde{I}(q)$ in Ref. 2 are related by $\tilde{I}(q) = mQ(q) - 1$ and hence $|Q(q)| = \frac{1}{m}|1 + \tilde{I}(q)| \leq$

$\frac{1}{m}(1 + |\tilde{I}(q)|)$.) To prove (20) for $m \geq 47$, we set $\delta \equiv 0.04$ and consider $q = 3, 5, \ldots 4m - 5$ in the regions $\frac{4}{\pi}\frac{\sqrt{m+1/2}}{q} \leq 1 - \delta$ and $\frac{4}{\pi}\frac{\sqrt{m+1/2}}{q} > 1 - \delta$ separately. In the former q-region, by (Ref. 2, (6.21)), $|1 + \tilde{I}(q)| \leq 1 + |\tilde{I}(q)| \leq 1.96$. In the latter q-region, substituting $\frac{q}{\sqrt{m+1/2}} \leq \frac{4}{\pi}\frac{1}{1-\delta}$ in (Ref. 2, (6.55)), we note that the resulting estimate on $|1 + \tilde{I}(q)|$ multiplied by $(\frac{1}{2} - \frac{(m!)^2}{m(2m)!} 2^{2m-2})$, is < 0.98 in fact for $m \geq 44$. These facts together with Lemma 2 prove (20) (cf. Ref. 2, (6.56), (6.57)).

It remains to prove Lemma 3. A straightforward computation using (15)(i) and (16) shows that u is a solution of the equation

$$x(1 - x^2)u' - 4mu^2 + (2(m+1)x^2 + 1 - 2m)u - \frac{x^2}{2m} = 0. \tag{21}$$

Moreover as $h(x) > 0$, u is smooth. By (15)(ii), and by differentiating (21), we find,

$$u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = -\frac{1}{m(2m-3)}$$

$$u(1) = \frac{1}{2m}, \qquad u'(1) = \frac{2}{3} + \frac{1}{3m}. \tag{22}$$

Now observe that at a point $0 < x < 1$ where $u'(x) = 0$, we cannot have $4m(m+1)u(x) - 1 = 0$, i.e. $u(x) = \frac{1}{4m(m+1)}$. Indeed, substituting these values into (21), we find $-1 + (1 - 2m)(m+1) = 0$, which is a contradiction. Next we show that

$$(u'(x) = 0 \text{ for some } 0 < x < 1) \Longrightarrow \quad u''(x) = \frac{(4m(m+1)u(x) - 1)^2}{m(1 - 2mu(x))(1 - 4mu(x))}. \tag{23}$$

Indeed, differentiating (21), we find for such a point x

$$u''(x) = \frac{1 - 4m(m+1)u(x)}{m(1 - x^2)}. \tag{24}$$

Setting $u'(x) = 0$ in (21) and solving for $(1 - x^2)$ in terms of $u(x)$, we obtain

$$1 - x^2 = -\frac{(1 - 2mu(x))(1 - 4mu(x))}{4m(m+1)u(x) - 1}. \tag{25}$$

Note that by the above argument, the denominator in (25) is non-zero: also the numerator is non-zero as $1 - x^2 \neq 0$. Substituting (25) into (24) we obtain (23). Furthermore, the calculation shows that if $u'(x) = 0$ for some $0 < x < 1$, then $u''(x)$ is (finite and) non-zero.

From (22) we see that for small $x > 0$, $u(x) < 0$. As $u(1) > 0$, there must be at least one point $x \in (0, 1)$ where $u(x) = 0$. But it follows from (21) that if

$u(x) = 0$, $x \in (0, 1)$, then $u'(x) = \frac{x}{2m(1-x^2)} > 0$. Hence u crosses the level zero at a unique point $x_1 \in (0, 1)$. Next suppose that $u'(\hat{x}) = 0$ for some $\hat{x} \in (0, x_1)$. But then by (23), $u''(\hat{x}) > 0$ as $u(\hat{x}) < 0$. Thus any critical point for $u(x)$ in $(0, x_1)$, must be a local minimum. As $u(x)$ clearly has a minimum on $(0, x_1)$, it follows that it has a unique minimum at $x_0 \in (0, x_1)$, say, and no other critical points on $(0, x_1)$. Thus $u'(x) < 0$ for $0 < x < x_0$, and $u'(x) > 0$ for $x_0 < x \le x_1$.

Next we show that

$$0 < u(x) < \frac{1}{2m} \quad \text{for } x_1 < x < 1. \tag{26}$$

Indeed, if $u(x) = \frac{1}{2m}$ for $0 < x < 1$, then from (21) we find $u'(x) = \frac{2m+1}{2mx} > 0$. But we know from (22) that $u(1) = \frac{1}{2m}$, $u'(1) > 0$. Hence $u(x)$ cannot cross the level $\frac{1}{2m}$ for $0 < x < 1$. This proves (26).

To complete the proof that u is unimodal we show that $u'(x) > 0$ for $x_1 < x < 1$. Suppose $u'(x_2) < 0$ for some $x_1 < x_2 < 1$. Then as $u(x_1) = 0$ and $u(x_2) < u(1) = \frac{1}{2m}$, there must exist $x_1 < x_2^- < x_2$ and $x_2 < x_2^+ < 1$ such that u has a local maximum at x_2^- and a local minimum at x_2^+. By (23), we must have $u(x_2^-) > \frac{1}{4m}$ and $u(x_2^+) < \frac{1}{4m}$. This implies, in particular, that $u(x)$ crosses the level $\frac{1}{4m}$ at at least one point $x^\# \in (x_2^-, x_2^+)$ such that $u'(x^\#) \le 0$. But by (21), $u(x) = \frac{1}{4m}$, $0 < x < 1$, implies $u'(x) = \frac{1}{2x} > 0$, which is a contradiction. Thus $u'(x) \ge 0$ on $(x_1, 1)$. On the other hand if $u'(x_3) = 0$ for some $x_1 < x_3 < 1$, then by (23), $u''(x_3) \ne 0$ and so $u'(x)$ changes sign in a neighborhood of x_3, contradicting $u'(x) \ge 0$ on $(x_1, 1)$. Thus $u'(x) > 0$ for all $x_1 \le x \le 1$. This completes, in particular, the proof of part (i) of Lemma 3.

It remains to show that $u(x) = u(x; m) > -\frac{1}{4m}$ for $m \ge 2$, $x \in [0, 1]$. It turns out that $x = x_m \equiv \sqrt{\frac{m-1}{m+2}}$ plays a distinguished role. More precisely, as we now show,

$$u(x_m) > -\frac{1}{4m} \quad \Longrightarrow \quad \left(u(x) > -\frac{1}{4m} \text{ for all } x \in [0, 1]\right). \tag{27}$$

To see this, suppose $u(x) = -\frac{1}{4m}$ for some $x \in (0, 1)$: then from (21) we obtain

$$u'(x) = \frac{(m+2)x^2 - (m-1)}{2mx(1-x^2)}. \tag{28}$$

Suppose $u(x_m) > -\frac{1}{4m}$. If $u(\hat{x}) \le -\frac{1}{4m}$ for some $0 < \hat{x} < x_m$, then clearly $u(x^\#) = -\frac{1}{4m}$, $u'(x^\#) \ge 0$ for some point $x^\# \in [\hat{x}, x_m)$. But by (28), $u'(x^\#) < 0$, which is a contradiction. Similarly if $u(\hat{x}) \le -\frac{1}{4m}$ for some $x_m < \hat{x} < 1$, there must exist a point $x^\# \in (x_m, \hat{x}]$ such that $u(x^\#) = -\frac{1}{4m}$, $u'(x^\#) \le 0$. But this contradicts (28) as above. This proves (27).

To complete the proof of Lemma 3, we must prove $u(x_m) \equiv u(x_m; m) > -\frac{1}{4m}$, $m \geq 2$. Set $s = 1 - x$. From 15(iii), we obtain

$$h(x) = \frac{4m(1-s)^{2m-1}}{\sqrt{s(2-s)}} \int_0^s \frac{(1-\tau)^{-2m}}{\sqrt{\tau(2-\tau)}}\, d\tau \leq \frac{4m(1-s)^{2m}}{(2-s)(1-s)\sqrt{s}} \int_0^s \frac{(1-\tau)^{-2m}}{\sqrt{\tau}}\, d\tau.$$

Using the elementary inequality $\frac{1-s}{1-\tau} \leq e^{\tau - s}$ for $0 \leq \tau \leq s \leq 1$, we find

$$h(x) \leq \frac{4me^{-2ms}}{(2-s)(1-s)\sqrt{s}} \int_0^s \frac{e^{2m\tau}}{\sqrt{\tau}}\, d\tau = \frac{4me^{-\mu^2}}{\left(1 - \frac{\mu^2}{4m}\right)\left(1 - \frac{\mu^2}{2m}\right)\mu} \int_0^\mu e^{\lambda^2}\, d\lambda$$

where

$$\mu = \sqrt{2ms} = \sqrt{2m(1-x)}.$$

In order to prove $u(x_m) > -\frac{1}{4m}$, $m \geq 2$, we see that it is sufficient to show that

$$\frac{\left(1 - \frac{\mu^2}{2m}\right)\mu e^{\mu^2}}{2\int_0^\mu e^{\lambda^2}\, d\lambda} - \mu^2 + \frac{1}{1 - \frac{\mu^2}{4m}} > 0 \qquad \text{for } \mu = \mu_m = \sqrt{2m(1-x_m)}.$$

By the inequality $(1 - \frac{\mu^2}{4m})^{-1} > 1 + \frac{\mu^2}{4m}$, and the elementary fact that $1 < \mu_m < \sqrt{3}$, $m \geq 2$, we see that it is sufficient to show

$$F(\mu) \geq \frac{1}{m} G(\mu) \qquad \text{for } 1 \leq \mu \leq \sqrt{3}$$

where

$$F(\mu) \equiv \mu e^{\mu^2} + 2(1-\mu^2)\int_0^\mu e^{\lambda^2}\, d\lambda, \qquad G(\mu) \equiv \frac{\mu^2}{2}\left(\mu e^{\mu^2} - \int_0^\mu e^{\lambda^2}\, d\lambda\right).$$

But $G(\mu)$ is clearly increasing and so it is enough to show

$$F(\mu) \geq \frac{G(\sqrt{3})}{m} \qquad \text{for } 1 \leq \mu \leq \sqrt{3}. \tag{29}$$

Differentiating $F(\mu)$ we find

$$F(1) = e, \quad F'(1) = 3e - 4\int_0^1 e^{\lambda^2}\, d\lambda > 2.304 > 0$$

$$F''(1) = 2e - 4\int_0^1 e^{\lambda^2}\, d\lambda > -0.415$$

$$F'''(\mu) \geq 0 \quad \text{for } \mu \geq 1.$$

Thus for $1 \leq \mu \leq \sqrt{3}$

$$F(\mu) \geq F(1) + F'(1)(\mu - 1) + \frac{F''(1)}{2}(\mu-1)^2 \geq e - \frac{0.415}{2}(\sqrt{3}-1)^2 > 2.607.$$

On the other hand $G(\sqrt{3}) < 41.3$, and if we choose m so that $2.607 > \frac{41.3}{m}$, then (29) will hold. Clearly $m \geq 16$ satisfies this inequality. We conclude that $u(x_m) > -\frac{1}{4m}$ for $m \geq 16$. On the other hand, using Maple (only sums and products are involved), we find from (1), (16)

$$\min_{2 \leq m \leq 15} \left(u(x_m) + \frac{1}{4m} \right) > 0.0129 > 0.$$

This completes the proof of Lemma 3, and hence Theorem 1.

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REFERENCES

1. M. Adler and P. van Moerbeke, Toda versus Pfaff lattice and related polynomials. *Duke Math. J.* **112**:1–58 (2002).
2. P. Deift and D. Gioev, *Universality in Random Matrix Theory for Orthogonal and Symplectic Ensembles*. *IMRP Int. Math. Res. Pap.* (in press), `math-ph/0411075`.
3. P. Deift and D. Gioev, Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices. *Comm. Pure Appl. Math.* (in press), `math-ph/0507023`.
4. P. Deift, D. Gioev, T. Kriecherbauer, and M. Vanlessen, *Universality for orthogonal and symplectic Laguerre-type ensembles* (submitted), `math-ph/0612007`.
5. P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.* **52**:1491–1552 (1999).
6. N. M. Ercolani and K. D. T.-R. McLaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *Int. Math. Res. Not.* **2003**:755–820 (2003).
7. K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91**:151–204 (1998).
8. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV*, Academic Press, New York, London (1978).

9. A. Stojanovic, Universality in orthogonal and symplectic invariant matrix models with quartic potential. *Math. Phys. Anal. Geom.* **3**:339–373 (2000). Errata: *ibid.* **7**:347–349 (2004).
10. C. A. Tracy and H. Widom, Correlation functions, cluster functions, and spacing distributions for random matrices. *J. Statist. Phys.* **92**:809–835 (1998).
11. H. Widom, On the relation between orthogonal, symplectic and unitary matrix ensembles. *J. Statist. Phys.* **94**:347–363 (1999).